

on the tangent line of the section. Hence from (5) or (6)

$$\frac{\cos i \delta \phi}{\cos i' \delta \phi} = \frac{\varpi^4}{\rho^4} \cdot \frac{mm'}{n} \cdot \frac{m'^3}{mn} = \frac{\varpi^4 m'^4}{n^2 \rho^4} = \frac{m^4 n^2}{\rho^4},$$

since $\varpi m' = mn$. This agrees with Mr. Stewart's result (p. 197), since the R_e and $\frac{1}{2}R$ of Mr. Stewart are the same as the R' and R of equation (3).

IV. "On the Intensity of the Light reflected from or transmitted through a Pile of Plates." By GEORGE G. STOKES, M.A., Sec. R.S., Lucasian Professor of Mathematics in the University of Cambridge. Received January 1, 1862.

The frequent employment of a pile of plates in experiments relating to polarization suggests, as a mathematical problem of some interest, the determination of the mode in which the intensity of the reflected light, and the intensity and degree of polarization of the transmitted light, are related to the number of the plates, and, in case they be not perfectly transparent, to their defect of transparency.

The plates are supposed to be bounded by parallel surfaces, and to be placed parallel to one another. They will also be supposed to be formed of the same material, and to be of equal thickness, except in the case of perfect transparency, in which case the thickness does not come into account. The plates themselves and the interposed plates of air will be supposed, as is usually the case, to be sufficiently thick to prevent the occurrence of the colours of thin plates, so that we shall have to deal with intensities only.

On account of the different proportions in which light is reflected at a single surface according as the light is polarized in or perpendicularly to the plane of incidence, we must take account separately of light polarized in these two ways. Also, since the rate at which light is absorbed varies with its refrangibility, we must take account separately of the different constituents of white light. If, however, the plates be perfectly transparent, we may treat white light as a whole, neglecting as insignificant the chromatic variations of reflecting power. Let ρ be the fraction of the incident light reflected at the first surface of a plate. Then $1-\rho$ may be taken as the intensity of

the transmitted light*. Also, since we know that light is reflected in the same proportion externally and internally at the two surfaces of a plate bounded by parallel surfaces, the same expressions ρ and $1-\rho$ will serve to denote the fractions reflected and transmitted at the second surface. We may calculate ρ in accordance with Fresnel's formulæ from the expressions

$$\sin i' = \frac{\sin i}{\mu}, \quad (1)$$

$$\rho = \frac{\sin^2(i-i')}{\sin^2(i+i')}, \text{ or } = \frac{\tan^2(i-i')}{\tan^2(i+i')}, \quad (2)$$

according as the light is polarized in or perpendicularly to the plane of incidence.

In the case of perfect transparency, we may in imagination make abstraction of the substance of the plates, and state the problem as follows:—There are $2m$ parallel surfaces (m being the number of plates) on which light is incident, and at each of which a given fraction ρ of the light incident upon it is reflected, the remainder being transmitted; it is required to determine the intensity of the light reflected from or transmitted through the system, taking account of the reflexions, infinite in number, which can occur in all possible ways.

This problem, the solution of which is of a simpler form than that of the general case of imperfect transparency, might be solved by a particular method. As, however, the solution is comprised in that of the problem which arises when the light is supposed to be partially absorbed, I shall at once pass on to the latter.

In consequence of absorption, let the intensity of light traversing a plate be reduced in the proportion of 1 to $1-qdx$ in passing over the elementary distance dx within the plate. Let T be the thickness of a plate, and therefore $T \sec i'$ the length of the path of the light within it. Then, putting for shortness

$$e^{-qT \sec i'} = g, \quad (3)$$

1 to g will be the proportion in which the intensity is reduced by

* In order that the intensity may be measured in this simple way, which saves trouble in the problem before us, we must define the intensity of the light transmitted across the first surface to mean what would be the intensity if the light were to emerge again into air across the second surface without suffering loss by absorption, or by reflexion at that surface.

absorption in a single transit. The light reflected by a plate will be made up of that which is reflected at the first surface, and that which suffers 1, 3, 5, &c. internal reflexions. If the intensity of the incident light be taken as unity, the intensities of these various portions will be

$$\rho, (1-\rho)^2 \rho g^2, (1-\rho)^2 \rho^3 g^4, \&c.$$

and if r be the intensity of the reflected light, we have, by summing a geometric series,

$$r = \rho + \frac{(1-\rho)^2 \rho g^2}{1-\rho^2 g^2} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Similarly, if t be the intensity of the transmitted light,

$$t = \frac{(1-\rho)^2 g}{1-\rho^2 g^2}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and we easily find

$$r = \rho + g \rho t; \quad r + t = 1 - \frac{(1-\rho)(1-g)}{1-\rho g},$$

which is in general less than 1, but becomes equal to 1 in the limiting case of perfect transparency, in which case $g=1$.

The values of μ , i , and q in any case being supposed known, the formulæ (1), (2), (3), (4), (5) determine r and t , which may now therefore be supposed known. The problem therefore is reduced to the following:—There are m parallel plates of which each reflects and transmits given fractions r , t of the light incident upon it: light of intensity unity being incident on the system, it is required to find the intensities of the reflected and refracted light.

Let these be denoted by $\phi(m)$, $\psi(m)$. Consider a system of $m+n$ plates, and imagine these grouped into two systems, of m and n plates respectively. The incident light being represented by unity, the light $\phi(m)$ will be reflected from the first group, and $\psi(m)$ will be transmitted. Of the latter the fraction $\psi(n)$ will be transmitted by the second group, and $\phi(n)$ reflected. Of the latter the fraction $\psi(m)$ will be transmitted by the first group, and $\phi(m)$ reflected, and so on. Hence we get for the light reflected by the whole system,

$$\phi(m) + (\psi m)^2 \phi(n) + (\psi m)^2 \phi(m) (\phi n)^2 + \dots,$$

and for the light transmitted,

$$\psi(m) \psi(n) + \psi(m) \phi(n) \phi(m) \psi(n) + \psi(m) (\phi n)^2 (\phi m)^2 \psi(n) + \dots,$$

which gives, by summing the two geometric series,

$$\phi(m+n) = \phi(m) + \frac{(\psi m)^2 \phi(n)}{1 - \phi(m)\phi(n)}; \quad . \quad . \quad . \quad (6)$$

$$\psi(m+n) = \frac{\psi(m)\psi(n)}{1 - \phi(m)\phi(n)}. \quad . \quad . \quad . \quad (7)$$

We get from (6)

$$\phi(m+n)\{1 - \phi(m)\phi(n)\} = \phi(m) + \phi(n)\{(\psi m)^2 - (\phi m)^2\};$$

and the first member of this equation being symmetrical with respect to m and n , we get, by interchanging m and n and equating the results,

$$\phi(m) + \phi(n)\{(\psi m)^2 - (\phi m)^2\} = \phi(n) + \phi(m)\{(\psi n)^2 - (\phi n)^2\};$$

or

$$\frac{1}{\phi(m)}\{1 + (\phi m)^2 - (\psi m)^2\} = \frac{1}{\phi(n)}\{1 + (\phi n)^2 - (\psi n)^2\},$$

which is therefore constant. Denoting this constant for convenience by $2 \cos \alpha$, we have

$$(\psi m)^2 = 1 - 2 \cos \alpha \cdot \phi(m) + (\phi m)^2. \quad . \quad . \quad . \quad (8)$$

Squaring (7), and eliminating the function ψ by means of (8), we find

$$\begin{aligned} &\{1 - \phi(m)\phi(n)\}^2 \{1 - 2 \cos \alpha \cdot \phi(m+n) + [\phi(m+n)]^2\} \\ &= \{1 - 2 \cos \alpha \cdot \phi(m) + (\phi m)^2\} \{1 - 2 \cos \alpha \cdot \phi(n) + (\phi n)^2\}. \end{aligned} \quad (9)$$

From the nature of the problem, m and n are positive integers, and it is only in that case that the functions ϕ , ψ , as hitherto defined, have any meaning. We may, however, contemplate functions ϕ , ψ of a continuously changing variable, which are defined by the equations (6) and (7); and it is evident that if we can find such functions, they will in the particular case of a positive integral value of the variable be the functions which we are seeking.

In order that equations (6), (7) may hold good for a value zero of one of the variables, suppose n , we must have $\phi(0)=0$, $\psi(0)=1$. The former of these equations reduces (9) for $n=0$ to an identical equation. Differentiating (9) with respect to n , and after differentiation putting $n=0$, we find

$$\begin{aligned} &\phi'(0)\phi(m)\{1 - 2 \cos \alpha \cdot \phi(m) + (\phi m)^2\} + \cos \alpha \cdot \phi'(m) - \phi(m)\phi'(m) \\ &= \cos \alpha \cdot \phi'(0)\{1 - 2 \cos \alpha \cdot \phi(m) + (\phi m)^2\}, \end{aligned}$$

or dividing out by $\phi(m) - \cos \alpha$, (for $\phi(m) = \cos \alpha$ would only lead to $\phi(m) = \cos \alpha = 0$, $\psi(m) = C$),

$$\phi'(m) = \phi'(0) \{1 - 2 \cos \alpha \phi(m) + (\psi(m))^2\}. \quad (10)$$

Integrating this equation, determining the arbitrary constant by the condition that $\phi(m) = 0$ when $m = 0$, and writing β for $\sin \alpha \cdot \phi'(0)$, we have

$$\phi(m) = \frac{\sin m\beta}{\sin(\alpha + m\beta)}. \quad (11)$$

Substituting in (8) and reducing, we find

$$(\psi(m))^2 = \frac{\sin^2 \alpha}{\sin^2(\alpha + m\beta)}. \quad (12)$$

But (8) was derived, not from (7) directly, but from (7) squared; and on extracting the square root of both sides of (12), we must choose that sign which shall satisfy (7), and therefore we must take the sign +, as we see at once on putting $m = n = 0$. The equation (12) on taking the proper root and (11) may be put under the form

$$\frac{\phi(m)}{\sin(m\beta)} = \frac{\psi(m)}{\sin \alpha} = \frac{1}{\sin(\alpha + m\beta)}; \quad (13)$$

and to determine the arbitrary constants α, β we have, putting $m = 1$, and $\phi(m) = r$, $\psi(m) = t$,

$$\frac{r}{\sin \beta} = \frac{t}{\sin \alpha} = \frac{1}{\sin(\alpha + \beta)}. \quad (14)$$

We readily get from equations (13),

$$1 - \phi(m)\phi(n) = \frac{\sin \alpha \sin \{\alpha + (m+n)\beta\}}{\sin(\alpha + m\beta) \sin(\alpha + n\beta)};$$

$$\phi(m+n) - \phi(m) = \frac{\sin \alpha \sin n\beta}{\sin \{\alpha + (m+n)\beta\} \sin(\alpha + m\beta)};$$

whence the equations (6), (7) are easily verified. This verification seems necessary in logical strictness, because we have no right to assume *à priori* that it is possible to satisfy (6) and (7) for general values of the variables; and in deriving the equation (10), the equations (6) and (7) were only assumed to hold good for general values of m and infinitely small values of n .

The equations (13), (14) give the following *quasi*-geometrical construction for solving the problem:—Construct a triangle of which

the sides represent in magnitude the intensity of the incident, reflected, and refracted light in the case of a single plate, and then, leaving the first side and the angle opposite to the third unchanged, multiply the angle opposite to the second by the number of plates; the sides of the new triangle will represent the corresponding intensities in the case of the system of plates. I say *quasi*-geometrical, because the construction cannot actually be effected, inasmuch as the first side of our triangle is greater than the sum of the two others, and the angles are imaginary.

To adapt the formulæ (13), (14) to numerical calculation, it will be convenient to get rid of the imaginary quantities. Putting

$$\sqrt{\{ (1+r+t)(1+r-t)(1+t-r)(1-r-t) \}} = \Delta, \quad (15)$$

we have by the common formulæ of trigonometry,

$$\cos \alpha = \frac{1+r^2-t^2}{2r}; \quad \sin \alpha = \frac{\pm \sqrt{-1}\Delta}{2r};$$

whence, putting

$$\frac{1}{2r}(1+r^2-t^2+\Delta) = a, \quad \dots \dots (16)$$

we have

$$e^{\sqrt{-1}\alpha} = \cos \alpha + \sqrt{-1} \sin \alpha = a^{\mp 1}.$$

It is a matter of indifference which sign be taken: choosing the under signs, we have

$$2r \sin \alpha = -\sqrt{-1}\Delta, \quad e^{\sqrt{-1}\alpha} = a.$$

We have also

$$\cos \beta = \frac{1+t^2-r^2}{2t}, \quad \sin \beta = \frac{r}{t} \sin \alpha = -\frac{\sqrt{-1}\Delta}{2t},$$

no fresh ambiguity of sign being introduced. Putting therefore

$$\frac{1}{2t}(1+t^2-r^2+\Delta) = b, \quad \dots \dots (17)$$

we have

$$e^{\sqrt{-1}\beta} = b;$$

and equations (13) now give

$$\frac{\phi(m)}{b^m - b^{-m}} = \frac{\psi(m)}{a - a^{-1}} = \frac{1}{ab^m - a^{-1}b^{-m}} \dots \dots (18)$$

In the case of perfect transparency these expressions take a sim-

pler form. If $r+t$ differ indefinitely little from 1, α and β will be indefinitely small. Making α and β indefinitely small in (13) and (14), and putting $1-r$ for t , we find

$$\frac{\phi(m)}{mr} = \frac{\psi(m)}{1-r} = \frac{1}{1+(m-1)r} \dots \dots (19)$$

In this case it is evident that each of the $2m$ reflecting surfaces might be regarded as a separate plate reflecting light in the proportion of ρ to 1, and therefore we ought also to have, writing $2m$ for m and ρ for r in the denominators of the equations (19),

$$\frac{\phi(m)}{2m\rho} = \frac{\psi(m)}{1-\rho} = \frac{1}{1+(2m-1)\rho} \dots \dots (20)$$

It is easy to verify that when $g=1$ (4) reduces (19) to (20).

The following Table gives the intensity of the light reflected from or transmitted through a pile of m plates for the values 1, 2, 4, 8, 16, 32, and ∞ of m , for three degrees of transparency, and for certain selected angles of incidence. The assumed refractive index μ is 1.52. $\delta=1-e^{-gT}$ is the loss by absorption in a single transit of a plate at a perpendicular incidence, so that $\delta=0$ corresponds to perfect transparency. The most interesting angles of incidence to select appeared to be zero and the polarizing angle $\varpi=\tan^{-1} \mu$; but in the case of perfect transparency the result has also been calculated for an angle of incidence a little (2°) greater than the polarizing angle. ϕ denotes the intensity of the reflected and ψ that of the transmitted light, the intensity of the incident light being taken at 1000. For oblique incidences it was necessary to distinguish between light polarized in and light polarized perpendicularly to the plane of incidence; the suffixes 1, 2 refer to these two kinds respectively. For oblique incidences a column is added giving the ratio of ψ_1 to ψ_2 , which may be taken as a measure of the defect of polarization of the transmitted light. No such column was required for $\delta=0$ and $i=\varpi$, because in this case $\psi_2=1000$.

* From a paper by M. Wild in Poggendorff's 'Annalen' [vol. ix. (1856) p. 240], I find that the formulæ for the particular case of perfect transparency have already been given by M. Neumann. His demonstration does not appear to have been published.

m	$\delta=0$										$\delta=\frac{1}{2}\sigma$										$\delta=\frac{1}{10}$									
	$i=0$					$i=\sigma+\frac{1}{2}\sigma$					$i=0$					$i=\sigma$					$i=0$					$i=\sigma$				
	$i=0$		$i=\sigma$		$i=\sigma+\frac{1}{2}\sigma$		$i=0$		$i=\sigma$		$i=0$		$i=\sigma$		$i=\sigma+\frac{1}{2}\sigma$		$i=0$		$i=\sigma$		$i=0$		$i=\sigma$		$i=\sigma+\frac{1}{2}\sigma$		$i=0$		$i=\sigma$	
	ϕ	ψ	ϕ_1	ψ_1	ϕ_1	ψ_1	ϕ_1	ψ_1	ϕ_1	ψ_1	ϕ	ψ	ϕ_1	ψ_1	ϕ_1	ψ_1	ϕ	ψ	ϕ_1	ψ_1	ϕ_1	ψ_1	ϕ_1	ψ_1	ϕ_1	ψ_1	ϕ_1	ψ_1	ϕ_1	ψ_1
1	82	918	271	729	300	700	1	999	701	$\frac{\psi_1}{\psi_2}$	80	900	265	711	976	728	74	826	245	639	881	725								
2	151	849	426	574	459	541	2	998	542		145	815	410	544	953	571	125	686	351	435	777	559								
4	262	738	598	402	628	372	4	996	373		244	679	555	355	908	391	185	479	427	215	604	357								
8	416	584	749	251	771	229	8	992	231		364	490	656	182	824	221	229	237	451	57	365	156								
16	587	413	856	144	870	130	16	984	132		464	276	695	58	679	086	243	59	453	4	133	030								
32	740	260	922	78	931	69	32	968	071		509	97	699	7	461	014	244	4	453	0	18	001								
∞	1000	0	1000	0	1000	0	1000	0	000		516	0	699	0	0	000	244	0	453	0	0	000								

The intensity of the light reflected from an infinite number of plates, as we see from (18), is a^{-1} ; and since a is changed into a^{-1} by changing the sign of α or of Δ ,

$$a^{-1} = \frac{1}{2r}(1 + r^2 - t^2 - \Delta), \quad . \quad . \quad . \quad . \quad (21)$$

which is equal to 1 in the case of perfect transparency. Accordingly a substance which is at the same time finely divided, so as to present numerous reflecting surfaces, and which is of such a nature as to be transparent in mass, is brilliantly white by reflected light,—for example snow, and colourless substances thrown down as precipitates in chemical processes.

The intensity of the light reflected from a pile consisting of an infinite number of similar plates falls off rapidly with the transparency of the material of which the plates are composed, especially at small incidence. Thus at a perpendicular incidence we see from the above Table that the reflected light is reduced to little more than one half when 2 per cent. is absorbed in a single transit, and to less than a quarter when 10 per cent. is absorbed.

With imperfectly transparent plates, little is gained by multiplying the plates beyond a very limited number, if the object be to obtain light, as bright as may be, polarized by reflexion. Thus the Table shows that 4 plates of the less defective kind reflect 79 per cent., and 4 plates of the more defective as much as 94 per cent., of the light that could be reflected by a greater number, whereas 4 plates of the perfectly transparent kind reflect only 60 per cent.

The Table shows that while the amount of light transmitted at the polarizing angle by a pile of a considerable number of plates is materially reduced by a defect of transparency, its state of polarization is somewhat improved. This result might be seen without calculation. For while no part of the transmitted light which is polarized perpendicularly to the plane of incidence underwent reflexion, a large part of the transmitted light polarized the other way was reflected an even number of times; and since the length of path of the light within the absorbing medium is necessarily increased by reflexion, it follows that a defect of transparency must operate more powerfully in reducing the intensity of light polarized in, than of light polarized perpendicularly to the plane of polarization. But the Table also shows that a far better result can be obtained, as to the perfection of the polari-

zation of the transmitted light, without any greater loss of illumination, by employing a larger number of plates of a more transparent kind.

Let us now confine our attention to perfectly transparent plates, and consider the manner in which the degree of polarization of the transmitted light varies with the angle of incidence.

The degree of polarization is expressed by the ratio of ψ_1 to ψ_2 , which for brevity will be denoted by χ . When $\chi=1$ there is no polarization; when $\chi=0$ the polarization is perfect, in a plane perpendicular to the plane of incidence. Now ψ (which is used to denote ψ_1 or ψ_2 as the case may be) is given in terms of ρ by one of the equations (20), and ρ is given in terms of $i-i'$ and $i+i'$ by Fresnel's formulæ (2). Put

$$i-i'=\theta, \quad i+i'=\sigma;$$

then, from (1),

$$\frac{di}{\tan i} = \frac{di'}{\tan i'} = \frac{d\theta}{\tan i - \tan i'} = \frac{d\sigma}{\tan i + \tan i'} = \cos i \cos i' d\omega, \text{ suppose,}$$

whence

$$d\theta = \sin \theta d\omega, \quad d\sigma = \sin \sigma d\omega; \quad . \quad . \quad . \quad (22)$$

and we see that i and ω increase together from $i=0$ to $i=\frac{\pi}{2}$. We

have also

$$\rho_1 = \frac{\sin^2 \theta}{\sin^2 \sigma}, \quad d\rho_1 = \frac{2 \sin \theta}{\sin^3 \sigma} (\sin \sigma \cos \theta d\theta - \sin \theta \cos \sigma d\sigma) = \frac{2 \sin^2 \theta}{\sin^2 \sigma} (\cos \theta - \cos \sigma) d\omega;$$

$$\begin{aligned} \rho_2 &= \frac{\tan^2 \theta}{\tan^2 \sigma}, \quad d\rho_2 = \frac{2 \tan \theta}{\tan^3 \sigma} (\tan \sigma \sec^2 \theta d\theta - \tan \theta \sec^2 \sigma d\sigma) \\ &= \frac{2 \sin^2 \theta \cos \sigma}{\cos^3 \theta \sin^2 \sigma} (\cos \sigma - \cos \theta) d\omega = -\frac{\cos \sigma}{\cos^3 \theta} d\rho_1. \end{aligned}$$

Now $\cos \theta - \cos \sigma$ or $2 \sin i \sin i'$ is positive; and $\cos \sigma$ is positive from $i=0$ to $i=\varpi$, and negative from $i=\varpi$ to $i=\frac{\pi}{2}$. But (20) shows that ψ decreases as ρ increases. From $i=0$ to $i=\varpi$, ρ_1 increases and ρ_2 decreases, and therefore ψ_1 decreases and ψ_2 increases, and therefore on both accounts χ decreases. When $i=\varpi$, $\frac{d\rho_1}{di}$ is still positive, and therefore $\frac{d\psi_1}{di}$ negative, but ψ_2 has its maximum value 1, so that on passing through the polarizing angle χ still decreases, or the polari-

zation improves. When the plates are very numerous, $\psi_2=1$ at the polarizing angle, and on both sides of it decreases rapidly, whereas ψ_1 , which is always small, suffers no particular change about the polarizing angle. Hence in this case χ must be a minimum a little beyond the polarizing angle. Let us then seek the angle of incidence which makes χ a minimum in the case of an arbitrary number of plates.

We have from (20) and (2),

$$\begin{aligned}\chi &= \frac{\sin^2 \sigma - \sin^2 \theta}{\sin^2 \sigma + (2m-1) \sin^2 \theta} \cdot \frac{\sin^2 \sigma \cos^2 \theta + (2m-1) \sin^2 \theta \cos^2 \sigma}{\sin^2 \sigma \cos^2 \theta - \sin^2 \theta \cos^2 \sigma} \\ &= \frac{\sin^2 \sigma \cos^2 \theta + (2m-1) \sin^2 \theta \cos^2 \sigma}{\sin^2 \sigma + (2m-1) \sin^2 \theta} = 1 - \frac{2m}{\operatorname{cosec}^2 \theta + (2m-1) \operatorname{cosec}^2 \sigma}. \quad (23)\end{aligned}$$

Hence χ is a minimum along with $\operatorname{cosec}^2 \theta + (2m-1) \operatorname{cosec}^2 \sigma$. Differentiating, and taking account of the formulæ (22), we find, to determine the angle of maximum polarization, the very simple equation

$$\cos \theta \sin^2 \sigma + (2m-1) \cos \sigma \sin^2 \theta = 0. \quad (24)$$

For any assumed value of i from ϖ to $\frac{\pi}{2}$, this equation gives at once the value of m , that is, the number of plates of which a pile must be composed in order that the assumed incidence may be that of maximum polarization of the transmitted light. The equation may be put under the form

$$2m-1 = -\frac{\tan \sigma}{\tan \theta} \cdot \frac{\sin \sigma}{\sin \theta} = \frac{1}{\sqrt{\rho_1 \rho_2}}.$$

Now we have seen that both ρ_1 and ρ_2 continually increase, and therefore m continually decreases, from $i=\varpi$ to $i=\frac{\pi}{2}$. At the first of these limits $\rho_2=0$, and therefore $m=\infty$. At the second $\rho_1=\rho_2=1$, and therefore $m=1$. Hence with a single plate the polarization of the transmitted light continually improves up to a grazing incidence, but with a pile of plates the polarization attains a maximum at an angle of incidence which approaches indefinitely to the polarizing angle as the number of plates is indefinitely increased.

Eliminating m from (23) and (24), we find

$$\chi = -\cos \theta \cos \sigma, \quad (25)$$

which determines for any pile χ_1 , the defect of maximum polarization of the transmitted light, in terms of the angle of incidence for

which the polarization is a maximum. We have, from (25), (22), and (24),

$d\chi_1 = (\sin^2 \theta \cos \sigma + \sin^2 \sigma \cos \theta) d\omega = -2(m-1) \cos \sigma \sin^2 \theta d\omega$,
and $\cos \sigma$ is negative. Hence χ_1 decreases as ω (and therefore i) decreases, or as m increases. For $m=1$, $i=\frac{\pi}{2}$ and $\chi_1=\mu^{-2}$; for $m=\infty$, $\cos \sigma=0$, and therefore $\chi_1=0$, or the maximum polarization tends indefinitely to become perfect as the number of plates is indefinitely increased.

For a given number of plates the angle of maximum polarization may be readily found from (24) by the method of trial and error. But for merely examining the progress of the functions, instead of tabulating i for assumed values of m , it will serve equally well to tabulate m for assumed values of i . The following Table gives for assumed angles of incidence, decreasing by 5° from 90° , the number of plates required to make these angles the angles of maximum polarization of the transmitted light, and the value of χ_1 , which determines the defect of polarization.

$i =$	90°	85°	80°	75°	70°	65°	60°	$56^\circ 40' (= \varpi)$
$m =$	1	1.330	1.944	2.913	4.921	9.775	30.372	∞
$\chi_1 =$.433	.422	.390	.337	.265	.177	.075	0

V. "On the Theory of the Polyhedra." By the Rev. T. P. KIRKMAN, M.A., F.R.S.

This is a revised version of a Paper having the same title, read on the 30th of May, 1861, of which an abstract has been already given at page 218.

January 30, 1862.

Major-General SABINE, R.A., President in the Chair.

The following communications were read:—

- I. "On the Calculus of Symbols."—Second Memoir. By W. H. L. RUSSELL, Esq., A.B. Communicated by ARTHUR CAYLEY, Esq. Received January 7, 1862.

(Abstract.)

This memoir is the continuation of one on the calculus of symbols